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EXTREMAL PROBLEMS AND ISOTROPIC POSITIONS OF CONVEX BODIES

BY

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ABSTRACT

Let K be a convex body in $\mathbb{R}^n$ and let $W_i(K)$, $i = 1, \dots, n-1$ be its quermassintegrals. We study minimization problems of the form $\min\{W_i(TK) | \, T \in \mathrm{SL}_n\}$ and show that bodies which appear as solutions of such problems satisfy isotropic conditions or even admit an isotropic characterization for appropriate measures. This shows that several well known positions of convex bodies which play an important role in the local theory may be described in terms of classical convexity as isotropic ones. We provide new applications of this point of view for the minimal mean width position.

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1. Introduction

Given a convex body K in $\mathbb{R}^n$ we consider the family $\{TK|\ T \in \mathrm{SL}_n\}$ of its *positions*. One of the main problems in the asymptotic theory of finite dimensional normed spaces is introducing the right position of the unit ball K_X of a space X. There exist many well-known positions which have been introduced and used for different purposes in this theory: John's position, the ℓ-position, M-positions are among them (see [MSch1], [Pi2] and [TJ] for a description and important applications). Because of the isomorphic nature of the results of the asymptotic theory, an isomorphic point of view dominates the study of these special positions as well. Even the definition of some of them (the M-position is such an example) is done in isomorphic form.

The purpose of this paper is to discuss the possibility of an isometric approach to these questions. The standard isotropic position of a convex body provides a good example for our point of view:

Let K be a convex body in $\mathbb{R}^n$ with centroid at the origin and volume equal to one. We say that K is in **isotropic position** if

$$\int_K \langle x,\theta\rangle^2 dx = L_K^2$$

for every $\theta \in S^{n-1}$. It is not hard to see that every body K of volume one has a position which is isotropic. Moreover, this position is uniquely determined up to an orthogonal transformation. Therefore, L_K is an affine invariant which is called the **isotropic constant** of K.

The isotropic position is well studied and has several connections with classical convexity problems (see [MP1]). In particular, the question if $L_K \leq c$ for some absolute positive constant and every body K is a major open problem. The starting point of our present discussion is the following remark:

FACT I: A body K is isotropic if and only if $\int_K |x|^2dx \leq \int_{TK} |x|^2dx$ for every $T \in \mathrm{SL}_n$, where $|\cdot|$ is the standard Euclidean norm.

The proof of the "if" part is given by a simple variational argument: If $T \in L(\mathbb{R}^n)$ and $\varepsilon > 0$ is small enough, then $(I+\varepsilon T)/[\det(I+\varepsilon T)]^{1/n}$ is volume preserving, therefore

$$(1) \qquad \int_K |x+\varepsilon Tx|^2dx \geq [\det(I+\varepsilon T)]^{2/n}\int_K |x|^2dx.$$

Writing $|x+\varepsilon Tx|^2 = |x|^2 + 2\varepsilon\langle x,Tx\rangle + O(\varepsilon^2)$ and

$$[\det(I+\varepsilon T)]^{2/n} = 1 + 2\varepsilon\frac{\mathrm{tr}T}{n} + O(\varepsilon^2),$$

and letting $\varepsilon \to 0^+$ we get

$$\int_K \langle x, Tx\rangle dx \geq \frac{\mathrm{tr}T}{n} \int_K |x|^2 dx, \tag{2}$$

and replacing T by $-T$ we see that there must be equality in (2) for every $T \in L(\mathbb{R}^n)$. This in turn implies that K is isotropic.

Starting with the functional $T \to f(TK) = \int_{TK} |x|^2 dx$ on SL_n we saw that its minimum is achieved on some *isotropic position* (for the Lebesgue measure on K). In this paper we show that this is a general scheme which produces isometric descriptions for many classical positions of the theory.

As a second example, we mention the *minimal surface area position*: Let K be a convex body, and write $\partial(K)$ for its surface area. We say that K has **minimal surface area** if $\partial(K) \leq \partial(TK)$ for every $T \in \mathrm{SL}_n$.

A characterization of the minimal surface area position was given by Petty ([Pe], see also [GP]):

FACT II: A convex body K has minimal surface area if and only if

$$\int_{S^{n-1}} \langle u, \theta\rangle^2 \sigma_K(du) = \frac{\partial(K)}{n} \tag{3}$$

for every $\theta \in S^{n-1}$, where σ_K is the area measure of K.

Recall that the area measure σ_K of K is defined on S^{n-1} by

$$\sigma_K(A) = \nu(\{x \in \mathrm{bd}(K)\colon \text{the outer normal to } K \text{ at } x \text{ is in } A\}), \tag{4}$$

where ν is the $(n-1)$-dimensional surface measure on K. The key point for the proof of the fact is the observation that, for every $T \in \mathrm{SL}_n$,

$$\partial((T^{-1})^*K) = \int_{S^{n-1}} |Tx| \sigma_K(dx). \tag{5}$$

Then, we employ a variational argument identical to the one used for Fact I. One can also check that the minimal surface position is unique up to orthogonal transformations (see [GP] for the details).

In view of the above result we give the following definition:

Definition: A Borel measure μ on S^{n-1} will be called **isotropic** if

$$\int_{S^{n-1}} \langle u, \theta\rangle^2 \mu(du) = \frac{\mu(S^{n-1})}{n} \tag{6}$$

for every $\theta \in S^{n-1}$.

In this terminology, a body K has minimal surface area if and only if its area measure is isotropic: The minimum of the functional $T \to \partial(TK)$, $T \in \mathrm{SL}_n$ is again achieved on an isotropic position (for the appropriate measure on the sphere).

Surface area is one of the quermassintegrals $W_i(K)$ of the body K (see Section 2 for notation and definitions). We consider the minimization problems

$$\min\{W_i(TK) \mid T \in \mathrm{SL}_n\}, \quad i = 1, \dots, n-1. \tag{7}$$

In every case, a necessary condition for the minimal position is that the corresponding mixed area measure $S_{n-i}(K, \cdot)$ should be isotropic (see Section 4). In particular, in Section 3 we find a necessary and sufficient condition for the minimal mean width position: a body K has minimal mean width if and only if

$$\int_{S^{n-1}} h_K(u)\langle u, \theta\rangle^2 \sigma(du)$$

does not depend on $\theta \in S^{n-1}$, where h_K is the support function of K and σ is the rotationally invariant probability measure on the sphere. In the symmetric case, using a classical estimate of Pisier [Pi1] (after work of Lewis [L] and Figiel and Tomczak-Jaegermann [FT]) we see that isotropicity of the measure $h_K d\sigma$ implies the inequality

$$\int_{S^{n-1}} h_K(u)\sigma(du) \leq c \log d(X_K, \ell_2^n) \left(\frac{|K|}{|D_n|}\right)^{1/n}. \tag{8}$$

In Section 5 we see the maximal volume ellipsoid position (John's position) as a solution of the problem

$$\min\{\|T\colon \ell_2^n \to X_K\| \mid T \in \mathrm{SL}_n\}. \tag{9}$$

Using the same general method we give a simple proof of John's theorem in its full strength. In our present setting, John's representation of the identity may be interpreted as an isotropic condition: a symmetric body K is in John's position if and only if there is an isotropic measure supported by its contact points with the inscribed ball.

Finally, in Section 6 we show that M-position may also be described in an isometric way. If $|K| = |D_n|$, we study the problem

$$\min\{|TK + D_n| \mid T \in \mathrm{SL}_n\} \tag{10}$$

and show that if K is a solution, then $K + D_n$ must have minimal surface area. In view of Petty's result, this opens the possibility of an isotropic M-position.

K. Ball [Ba1,2,3] realized that John's representation of the identity could be combined with the Brascamp–Lieb inequality. This led him to sharp bounds for the volume ratio, the volume of the central sections of the cube, and an exact reverse isoperimetric inequality. The reverse Brascamp–Lieb inequality [Bar] has been recently applied for an estimate of the volume of the central sections of the difference body of a non-symmetric body [Ru]. Petty's isotropic description of the minimal surface area position (combined with the Brascamp–Lieb inequality) leads to sharp inequalities for the volume of the projection body and its polar in terms of the minimal surface parameter [GP]. All these results show that the general isotropic point of view we propose in this paper might help towards a new understanding of several isomorphic results of the theory.

2. Definitions and preliminaries

We first recall some facts about mixed volumes and mixed area measures. For detailed proofs we refer the reader to [Sch].

2.1. Let $\mathcal{K}_n$ denote the set of all non-empty, compact convex subsets of $\mathbb{R}^n$. We may view $\mathcal{K}_n$ as a convex cone under Minkowski addition and multiplication by nonnegative real numbers. Minkowski's theorem (and the definition of the *mixed volumes*) asserts that if $K_1, \dots, K_m \in \mathcal{K}_n, m \in \mathbf{N}$, then the volume of $t_1K_1 + \cdots + t_mK_m$ is a homogeneous polynomial of degree n in $t_i > 0$. That is,

$$(1) \qquad |t_1K_1 + \cdots + t_mK_m| = \sum_{1 \leq i_1, \dots, i_n \leq m} V(K_{i_1}, \dots, K_{i_n}) t_{i_1} \cdots t_{i_n},$$

where the coefficients $V(K_{i_1}, \dots, K_{i_n})$ are chosen to be invariant under permutations of their arguments. The coefficient $V(K_1, \dots, K_n)$ is called the mixed volume of $K_1, \dots, K_n$.

2.2. Steiner's formula may be seen as a special case of Minkowski's theorem. The volume of $K + tD_n$, $t > 0$, can be expanded as a polynomial in t:

$$(2) \qquad |K + tD_n| = \sum_{i=0}^{n} \binom{n}{i} W_i(K) t^i,$$

where $W_i(K) = V(K; n-i, D_n; i)$ is the *i-th quermassintegral* of K. Here and elsewhere we use the notation $L; j$ for $L, \dots, L$ j-times. The quermassintegrals inherit properties of mixed volumes: they are monotone, continuous with respect to the Hausdorff metric, and homogeneous of degree $n - i$.

2.3. The *mixed area measures* were introduced by Alexandrov [Al1,2] and may be viewed as a local generalization of the mixed volumes. For any $(n-1)$-tuple $\mathcal{C} = K_1, \dots, K_{n-1} \in \mathcal{K}_n$, the Riesz representation theorem guarantees the existence of a Borel measure $S(\mathcal{C}, \cdot)$ on the unit sphere S^{n-1} such that

$$V(L, K_1, \dots, K_{n-1}) = \frac{1}{n} \int_{S^{n-1}} h_L(u) dS(\mathcal{C}, u) \tag{3}$$

for every $L \in \mathcal{K}_n$, where h_L is the *support function* of L. The local analogue of Minkowski's theorem is

$$S_{n-1}\Big(\sum_{i=1}^{m} t_i K_i, \omega\Big) = \sum_{1 \le i_1, \dots, i_n \le m} S(K_{i_1}, \dots, K_{i_{n-1}}, \omega) t_{i_1} \cdots t_{i_{n-1}} \tag{4}$$

for all Borel $\omega \subseteq S^{n-1}$, $t_i > 0, K_i \in \mathcal{K}_n$, $m \in \mathbf{N}$ (see below for the definition of S_{n-1}).

The *j-th area measure* of K is defined by $S_j(K, \cdot) = S(K; j, D_n; n-j-1, \cdot)$, $j = 0, 1, \dots, n-1$. It follows that the quermassintegrals of K can be represented by

$$W_i(K) = \frac{1}{n} \int_{S^{n-1}} h_K(u) dS_{n-i-1}(K, u), \quad i = 0, 1, \dots, n-1 \tag{5}$$

or, alternatively,

$$W_i(K) = \frac{1}{n} \int_{S^{n-1}} dS_{n-i}(K, u), \quad i = 1, \dots, n. \tag{6}$$

2.4. Let $K_i \in \mathcal{K}_n$ and assume for simplicity that h_{K_i} is twice continuously differentiable. Then, the mixed area measure of $K_1, \dots, K_{n-1}$ has a continuous density $s(K_1, \dots, K_{n-1}, \cdot)$ with respect to the Lebesgue measure on S^{n-1}, the mixed discriminant of the second differentials of h_{K_i}. We write $s_j(K, u)$ for $s(K; j, D_n; n-j-1, u)$. It follows that

$$\int_{S^{n-1}} h_{K_1}(u) s(K_2, K_3, \dots, K_n, u) du = \int_{S^{n-1}} h_{K_2}(u) s(K_1, K_3, \dots, K_n, u) du. \tag{7}$$

In particular, for $i = 1, \dots, n-1$ we have

$$W_i(K) = \frac{1}{n} \int_{S^{n-1}} s_{n-i}(K, u) du = \frac{1}{n} \int_{S^{n-1}} h_K(u) s_{n-i-1}(K, u) du. \tag{8}$$

2.5. Let f be a real function on $\mathbb{R}^n\backslash\{o\}$. We write $\hat{f}$ for the restriction of f to S^{n-1}. If F is defined on S^{n-1}, the radial extension f of F to $\mathbb{R}^n\backslash\{o\}$ is given by $f(x) = F(x/|x|)$. If F is a twice differentiable function on S^{n-1}, we define

$$\Delta_o F = (\hat{\Delta f}) \quad \text{and} \quad \nabla_o F = (\hat{\nabla f}), \tag{9}$$

where f is the radial extension of F. The operator Δ_o is usually called the *Laplace–Beltrami operator*, while ∇_o is referred to as the *gradient*. As a consequence of Green's formula we have

$$\int_{S^{n-1}} F\Delta_o G = \int_{S^{n-1}} G\Delta_o F = -\int_{S^{n-1}} (\nabla_o F)\cdot(\nabla_o G). \tag{10}$$

For more details we refer the reader to [Gr].

2.6. If K is an origin symmetric convex body in $\mathbb{R}^n$, then K induces a norm $\|\cdot\|_K$ on $\mathbb{R}^n$ in a natural way. We shall write X_K for the normed space with unit ball K, and K_X for the unit ball of X. The polar body of K is defined by $\|x\|_{K^\circ} = \max_{y\in K}|\langle x,y\rangle| = h_K(x)$, and will be denoted by K°.

We consider the average

$$M(K) = \int_{S^{n-1}} \|x\|_K \sigma(dx) \tag{11}$$

of the norm $\|\cdot\|_K$ on S^{n-1}, and define $M^*(K) = M(K^\circ)$.

If K and L are bodies in $\mathbb{R}^n$, their multiplicative distance $d(K,L)$ is defined by

$$d(K,L) = \inf\{ab : a,b > 0, K \subseteq bL, L \subseteq aK\}. \tag{12}$$

The Banach–Mazur distance between X_K and X_L is

$$d(X_K, X_L) = \inf\{d(K,TL) |\, T \in \mathrm{GL}_n\}. \tag{13}$$

Whenever we write $(1/a)|x| \leq \|x\|_K \leq b|x|$, we assume that a,b are the smallest positive numbers for which this inequality holds true for every $x \in \mathbb{R}^n$. In particular, we then have $d(K, D_n) = ab$.

Finally, we denote by $G_{n,k}$ the Grassmannian of all k-dimensional subspaces of $\mathbb{R}^n$, equipped with the Haar probability measure $\nu_{n,k}$. We write $|K|$ for the volume of K, and ω_n for the volume of the Euclidean unit ball. The letters c, c', C etc. are reserved for absolute positive constants.

3. Minimal mean width

Let K be a convex body in $\mathbb{R}^n$ (without loss of generality we may assume that $o \in \mathrm{int} K$). The **mean width** $w(K)$ of K is the quantity

$$w(K) = 2\int_{S^{n-1}} h_K(u)\sigma(du). \tag{1}$$

This is equal to $2M^*(K)$ in the symmetric case. From 2.3 we see that

$$W_{n-1}(K) = \frac{1}{n}\int_{S^{n-1}} h_K(u)du = \omega_n \int_{S^{n-1}} h_K(u)\sigma(du), \tag{2}$$

hence,

$$w(K) = \frac{2W_{n-1}(K)}{\omega_n}. \tag{3}$$

We say that K has **minimal mean width** if $w(K) \leq w(TK)$ for every $T \in \mathrm{SL}_n$. This notion was heavily used in the literature under a different name: K has minimal mean width if and only if the ℓ-ellipsoid of K° is a multiple of D_n [FT]. Our purpose is to find necessary and sufficient conditions for a body K to have minimal mean width. We assume for simplicity that h_K is twice continuously differentiable (we then say that K is **smooth enough**).

THEOREM 3.1: *A smooth enough convex body K in $\mathbb{R}^n$ has minimal mean width if and only if*

$$2\int_{S^{n-1}} \langle \nabla h_K(u), Tu\rangle \sigma(du) = \frac{\mathrm{tr} T}{n} w(K) \tag{4}$$

for every $T \in L(\mathbb{R}^n)$. Moreover, this minimal mean width position is unique up to an orthogonal transformation.

Proof: Assume first that K has minimal mean width. Let $T \in L(\mathbb{R}^n)$ and $\varepsilon > 0$ be small enough. Then $(I+\varepsilon T)^*/[\det(I+\varepsilon T)]^{1/n}$ is volume preserving, and this means that

$$\int_{S^{n-1}} h_K(u+\varepsilon Tu)\sigma(du) \geq [\det(I+\varepsilon T)]^{1/n} \int_{S^{n-1}} h_K(u)\sigma(du). \tag{5}$$

Since $h_K(u+\varepsilon Tu) = h_K(u) + \varepsilon\langle \nabla h_K(u), Tu\rangle + O(\varepsilon^2)$ and

$$[\det(I+\varepsilon T)]^{1/n} = 1 + \varepsilon\frac{\mathrm{tr} T}{n} + O(\varepsilon^2),$$

letting $\varepsilon \to 0^+$ we obtain

$$2\int_{S^{n-1}} \langle \nabla h_K(u), Tu\rangle \sigma(du) \geq \frac{\mathrm{tr} T}{n} w(K). \tag{6}$$

Replacing T by $-T$ in (6) we see that there must be equality in (4) for every $T \in L(\mathbb{R}^n)$.

Conversely, assume that (4) is satisfied and let $T \in \mathrm{SL}_n$. Up to an orthogonal transformation we may assume that T is symmetric positive-definite. Then,

$$w(TK) = 2\int_{S^{n-1}} h_{TK}(u)\sigma(du) = 2\int_{S^{n-1}} h_K(T^*u)\sigma(du). \tag{7}$$

It is a known fact that $\nabla h_K(u)$ is the unique point on the boundary of K at which u is the outer normal to K (see [Sch], p. 40). In particular, $\nabla h_K(u) \in K$, which implies

$$\langle \nabla h_K(u), z\rangle \leq h_K(z) \tag{8}$$

for every $z \in \mathbb{R}^n$. Therefore, by (7), (8) and (4) we get

$$w(TK) \geq 2\int_{S^{n-1}} \langle \nabla h_K(u), T^*u\rangle \sigma(du) = \frac{\mathrm{tr}T^*}{n} w(K) \geq w(K). \tag{9}$$

This shows that K has minimal mean width. Moreover, we can have equality in (9) only if T is the identity. This proves uniqueness of the minimal mean width position up to $U \in O(n)$. ∎

Consider the measure ν_K on S^{n-1} with density h_K with respect to σ. We shall prove that a smooth enough convex body K has minimal mean width if and only if ν_K is isotropic.

Lemma 3.2: *Let K be a smooth enough convex body in $\mathbb{R}^n$. We define*

$$I_K(\theta) = \int_{S^{n-1}} \langle \nabla h_K(u), \theta\rangle\langle u, \theta\rangle \sigma(du), \quad \theta \in S^{n-1}. \tag{10}$$

Then,

$$\frac{w(K)}{2} + I_K(\theta) = (n+1)\int_{S^{n-1}} h_K(u)\langle u, \theta\rangle^2 \sigma(du) \tag{11}$$

for every $\theta \in S^{n-1}$.

Proof: Let $\theta \in S^{n-1}$, and consider the function $f(x) = \langle x, \theta\rangle^2/2$. A direct computation shows that

$$(\nabla_\circ \hat{f})(u) = \langle u, \theta\rangle\theta - \langle u, \theta\rangle^2 u \tag{12}$$

and

$$(\Delta_\circ \hat{f})(u) = 1 - n\langle u, \theta\rangle^2. \tag{13}$$

Since h_K is positively homogeneous of degree 1, we have

$$(\nabla_\circ \hat{h_K})(u) = \nabla h_K(u) - h_K(u)u \quad \text{and} \quad h_K(u) = \langle \nabla h_K(u), u\rangle, u \in S^{n-1}.$$

Taking into account (12) we obtain

$$\langle(\nabla_\circ \hat{f})(u), (\nabla_\circ \hat{h_K})(u)\rangle = \langle \nabla h_K(u), \theta\rangle\langle u, \theta\rangle - h_K(u)\langle u, \theta\rangle^2. \tag{14}$$

Integrating on the sphere and using Green's formula (see 2.5), we have

$$I_K(\theta) - \int_{S^{n-1}} h_K(u)\langle u, \theta\rangle^2 \sigma(du) = -\int_{S^{n-1}} h_K(u)(\Delta_\circ \hat{f})(u)\sigma(du), \tag{15}$$

which is equal to

$$-\frac{w(K)}{2} + n\int_{S^{n-1}} h_K(u)\langle u, \theta\rangle^2 \sigma(du)$$

by (13). This proves (11). ∎

THEOREM 3.3: *A smooth enough convex body K has minimal mean width if and only if*

$$\int_{S^{n-1}} h_K(u)\langle u, \theta\rangle^2 \sigma(du) = \frac{w(K)}{2n} \tag{16}$$

for every $\theta \in S^{n-1}$ (equivalently, if ν_K is isotropic).

Proof: It is not hard to check that (4) is true for every $T \in L(\mathbb{R}^n)$ if and only if

$$I_K(\theta) = \frac{w(K)}{2n} \tag{17}$$

for every $\theta \in S^{n-1}$. The result now follows from Theorem 3.1 and Lemma 3.2. ∎

Remark: The smoothness assumption in Theorem 3.3 is not really needed. Assume for example that K is any convex body for which ν_K is isotropic. Given $\varepsilon > 0$, we may approximate K by a smooth body K_ε so that $I_{K_\varepsilon}(\theta)$ is up to ε constant on S^{n-1}. If $T_\varepsilon(K_\varepsilon)$ has minimal mean width for some symmetric and positive $T_\varepsilon \in \mathrm{SL}_n$, we easily check from (9) that $\mathrm{tr}T \leq (1 + O(\varepsilon))n$, and the stability of the arithmetic-geometric means inequality implies that T_ε is close to the identity. Passing to the limit as $\varepsilon \to 0^+$ and taking into account the fact that $T_\varepsilon(K_\varepsilon)$ has minimal mean width, we see that K has the same property. The other direction can be treated in a similar way.

The fact that (4) and (16) are linear in K has the following immediate consequence:

COROLLARY 3.4: *Let K_1 and K_2 be smooth enough convex bodies in $\mathbb{R}^n$.*

(i) *If K_1 and K_2 have minimal mean width, then their Minkowski sum K_1+K_2 has also minimal mean width.*

(ii) *If K_1 and K_1+K_2 have minimal mean width, then K_2 has also minimal mean width.*

Proof: Obvious from Theorem 3.1 or 3.3, since $h_{K_1+K_2} = h_{K_1} + h_{K_2}$ and $w(K_1+K_2) = w(K_1) + w(K_2)$. ∎

In the symmetric case, it is a well-known fact [L], [FT], [Pi1] that if W has minimal mean width then $M(W)M^*(W) \leq c\log d(X_W, \ell_2^n)$. As an application of this estimate and of Corollary 3.4 we obtain:

THEOREM 3.5: *Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$ and assume that its unit ball K has the property $M(K) \leq M(TK)$ for every $T \in \mathrm{SL}_n$. Then, for every $\lambda \in (0,1)$, there exists a $[(1-\lambda)n]$-dimensional section $K \cap E$ of K such that*

$$d(K\cap E, D_n \cap E) \leq c\frac{b}{M\sqrt{\lambda}}\log\left(\frac{2b}{M\sqrt{\lambda}}\right), \tag{18}$$

where $c > 0$ is an absolute constant.

Proof: Without loss of generality we may assume that $M(K) = 1$ and $\lambda < 1/2$. Let t_0 be the smallest integer t for which $\log^{(t)}(bM^*(K)) \leq 2$ (where $\log^{(t)}$ denotes the t-th iterated logarithm). The Low M^*-estimate [M1], [PT], [Go] implies that, for some absolute constant $\delta > 0$,

$$\|x\| \geq \frac{\delta\sqrt{\lambda/2^{t_0}}}{M^*(K)}|x| \tag{19}$$

for all $x \in E_0$ or $x \in E_0^{\perp}$, where E_0 is in a subset L_0 of $G_{n,[(1-2^{-t_0}\lambda)n]}$ of measure greater than $p(\lambda, n, t_0) = 1 - c_1\exp(-c_2 2^{-t_0}\lambda n)$, and $c_1, c_2 > 0$ are absolute constants.

Consider the orthogonal transformation $U = U(E_0) = P_{E_0} - P_{E_0^{\perp}}$, $E_0 \in L_0$. Then,

$$\frac{\|x\| + \|Ux\|}{2} \geq \frac{\delta\sqrt{\lambda/2^{t_0}}}{\sqrt{2}M^*(K)}|x| \tag{20}$$

for all $x \in \mathbb{R}^n$. Define a new body $K_1 = K_1(E_0)$ by $K_1^{\circ} = (K^{\circ} + U^*K^{\circ})/2$. Then, by Corollary 3.4, K_1° has minimal mean width equal to $M(K_1) = 1$. It follows that

$$M^*(K_1) = M(K_1)M^*(K_1) \leq c\log\left(\frac{\sqrt{2^{t_0+1}}M^*(K)b}{\delta\sqrt{\lambda}}\right). \tag{21}$$

Observe that $\|x\|_K = \|x\|_{K_1(E_0)}$ on E_0, for every $E_0 \in L_0$.

We now iterate this step: assume that $L_i \subset G_{n,[(1-2^{-t_0+i}\lambda)n]}$, $E_i \in L_i$, and $K_{i+1}(E_0, \dots, E_i)$, $i = 0, \dots, s-1$ have been defined and satisfy the following:

(i) $(K_{i+1})^\circ$ has minimal mean width, and $M(K_{i+1}) = 1$.

(ii) $M^*(K_{i+1}) \leq c \log(\sqrt{2^{t_0-i+1}} M^*(K_i) b/\delta\sqrt{\lambda})$.

(iii) $\|x\|_{K_{i+1}} = \|x\|_{K_i} = \cdots = \|x\|$, for all $x \in F_i = E_0 \cap \cdots \cap E_i$.

We apply the Low M^*-estimate to K_s, and find $L_s \subset G_{n,[(1-2^{-t_0+s}\lambda)n]}$ with measure $p(\lambda, n, t_0 - s)$ such that

$$b|x| \geq \|x\|_{K_s} \geq \frac{\delta\sqrt{\lambda/2^{t_0-s}}}{M^*(K_s)}|x| \tag{22}$$

on E_s and on $E_s^\perp$, for every $E_s \in L_s$. If $E_s \in L_s$, we define K_{s+1} by $K_{s+1}^\circ = (K_s^\circ + U^*(E_s)K_s^\circ)/2$. Then,

$$b|x| \geq \|x\|_{K_{s+1}} \geq \frac{\delta\sqrt{\lambda/2^{t_0-s}}}{\sqrt{2}M^*(K_s)}|x| \tag{23}$$

on $\mathbb{R}^n$, and

$$\|x\|_{K_{s+1}} = \|x\|_{K_s} = \cdots = \|x\| \tag{24}$$

for every $x \in F_s = E_0 \cap \cdots \cap E_s$. This means that

$$d(K \cap F_s, D_n \cap F_s) \leq \delta\sqrt{2^{t_0-s+1}/\lambda}\, b M^*(K_s). \tag{25}$$

We stop the procedure when $s = t_0$. Note that if $(E_0, E_1, \dots, E_{t_0})$ is a sequence as above, we have $\dim F_{t_0} \geq (1-\lambda)n$. Also, since each $(K_s)^\circ$ has minimal mean width, exactly as in (21) we get

$$M^*(K_{s+1}) \leq c \log\left(\frac{\sqrt{2^{t_0-s+1}} M^*(K_s) b}{\delta\sqrt{\lambda}}\right), \tag{26}$$

and this implies that $M^*(K_{t_0}) \leq C \log(b/\sqrt{\lambda})$. By (25), we have

$$d(K \cap F_{t_0}, D_n \cap F_{t_0}) \leq c\frac{b}{\sqrt{\lambda}} \log\left(\frac{b}{\lambda}\right). \qquad \blacksquare$$

Theorem 3.5 should be compared to an analogous result for the M-position: In [MSch2] it is proved that if K is in M-position of order $\alpha > 1/2$, and if there exist t orthogonal transformations $U_1, \dots, U_t$ such that $\frac{1}{t}\sum_{i=1}^t U_i K^\circ$ is c-equivalent to a ball, then for every $\lambda \in (0,1)$ there exists a subspace $F \in G_{n,(1-\lambda)n}$ such that $d(K \cap F, D_n \cap F) \leq C(t, \lambda, c)$. We can now show that the same is true for the minimal mean width position:

COROLLARY 3.6: *Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$ and assume that its unit ball K has the property $M(K) \leq M(TK)$ for every $T \in \mathrm{SL}_n$. Assume further that for some t orthogonal transformations $U_1, \dots, U_t$ and for some $0 < r, C < \infty$,*

$$r|x| \leq \frac{1}{t}\sum_{i=1}^{t} \|U_i x\| \leq Cr|x| \tag{27}$$

for all $x \in \mathbb{R}^n$. Then, for every $\lambda \in (0,1)$, there exists a $[(1-\lambda)n]$-dimensional section $K \cap E$ of K such that

$$d(K \cap E, D_n \cap E) \leq cC\frac{\sqrt{t}}{\sqrt{\lambda}}\log\left(\frac{2C\sqrt{t}}{\sqrt{\lambda}}\right). \tag{28}$$

Proof: Lemma 2.1 from [MSch2] and (27) imply that

$$b(K) = \max_{x \in S^{n-1}} \|x\| \leq Cr\sqrt{t}. \tag{29}$$

Since $M(K) \geq r$, we have

$$\frac{b(K)}{M(K)} \leq C\sqrt{t}. \tag{30}$$

The result is now a consequence of Theorem 3.5. ∎

An inspection of the argument we used for Theorem 3.5 shows that the statement holds true for a random $[(1-\lambda)n]$-dimensional section of K. This allows a "global" reformulation of Corollary 3.6:

COROLLARY 3.7: *With the same hypotheses as in Corollary 3.6, there exists one orthogonal transformation U such that, for some $r' > 0$,*

$$r'|x| \leq \|x\| + \|Ux\| \leq r'C\sqrt{t}\log(2C\sqrt{t})|x| \tag{31}$$

for all $x \in \mathbb{R}^n$. ∎

The example of $X = \ell_1^{n/10} \oplus \ell_\infty^{9n/10}$ from [MSch2] shows that such a statement cannot hold in general.

Let $t(K)$ be the smallest integer t for which there exist orthogonal transformations $U_1, \dots, U_t$ such that

$$\frac{M(K)}{2}|x| \leq \frac{1}{t}\sum_{i=1}^{t} \|U_i x\| \leq 2M(K)|x|$$

for all $x \in \mathbb{R}^n$. In [MSch2] it is shown that $t(K) \simeq (b/M(K))^2$. We will prove below an "isomorphic" version of this fact for bodies in ℓ-position. We fix $s \in$

$\{2,\dots,t(K)\}$ and ask how close to Euclidean can a norm $\|x\|_s = \frac{1}{s}\sum_{i=1}^{s}\|U_i x\|$, $U_i \in O(n)$ be. More precisely, let $g_K(s)$ be the smallest $A > 0$ for which there exist $r > 0$, $m \leq s$, and $U_1,\dots,U_m \in O(n)$ satisfying

$$r|x| \leq \frac{1}{m}\sum_{i=1}^{m}\|U_i x\| \leq rA|x|, \quad x \in \mathbb{R}^n.$$

From Lemma 2.1 in [MSch2] (see also the proof of Corollary 3.6), we must have $b(K) \leq rA\sqrt{m} \leq M(K)A\sqrt{s}$. This shows that

$$g_K(s) \geq c\sqrt{t(K)/s}. \tag{32}$$

We shall show that if K° has minimal mean width (if K has *minimal* M), then this estimate is sharp:

THEOREM 3.8: *Let* $\|\cdot\|$ *be a norm on* $\mathbb{R}^n$ *such that its unit ball* K *satisfies* $M(K) \leq M(TK)$, $T \in \mathrm{SL}_n$. *Then,*

$$c_1\sqrt{\frac{t(K)}{s}} \leq g_K(s) \leq c_2\sqrt{\frac{t(K)}{s}}\log\left(\frac{2t(K)}{s}\right), \tag{33}$$

where $c_1, c_2 > 0$ *are absolute constants.*

Proof: Let $s \in \{2,\dots,t(K)\}$, and set $b = b(K)$, $M = M(K)$. Following the proof of Theorem 2 in [BLM], one can check that there exist $s_1 = [s/2]$ and $U_1,\dots,U_{s_1} \in O(n)$ such that

$$\|x\|_{s_1} := \frac{1}{s_1}\sum_{i=1}^{s_1}\|U_i x\| \leq c\frac{b}{\sqrt{s_1}}|x| \leq c'\frac{b}{\sqrt{s}}|x| \tag{34}$$

for all $x \in \mathbb{R}^n$. Let K_1 be the unit ball of $\|\cdot\|_{s_1}$ and set $b_1 = b(K_1)$, $M_1 = M(K_1)$. Since $M_1 = M$, (34) implies that

$$t(K_1) \leq c'' t(K)/s. \tag{35}$$

Observe that K_1 has minimal M, therefore we can apply Corollary 3.7 with $C = 4$ and $t = t(K_1)$ to find $r > 0$ and $V \in O(n)$ such that

$$r|x| \leq \|x\|_{s_1} + \|Vx\|_{s_1} \leq c''' r\sqrt{t(K_1)}\log(2t(K_1))|x| \tag{36}$$

for all $x \in \mathbb{R}^n$. Setting $U_{s_1+i} = U_i V$, $i = 1,\dots,s_1$, and taking into account (35) we conclude the proof. ∎

Remark: Let $k(K)$ be the largest integer k for which a random k-dimensional central section of K is 4-equivalent to Euclidean. In [Msch2] it is proved that

$$\frac{1}{C}n \leq t(K)k(K) \leq Cn,$$

where $C > 0$ is an absolute constant. Having this duality in mind, one may view Theorem 3.8 as a global analogue (for bodies with minimal M) of the isomorphic version of Dvoretzky's theorem proved in [MSch3] (see also [GGM]): There exists a constant $c > 0$ such that, for every $k \geq c\log n$, every n-dimensional space K has a k-dimensional subspace F with $d(F, \ell_2^k) \leq c\sqrt{k/\log(n/k)}$.

Let us also mention the following common property of the M-position and the minimal mean width position: If both a/M^* and b/M are bounded by some constant C, then the space is $f(C)$-isomorphic to ℓ_2^n. This is proved in [MSch2] for the M-position, and follows from Pisier's inequality

$$MM^* \leq c\log(ab) \leq c\log(C^2MM^*) \tag{37}$$

for the minimal mean width position. The space $X = \ell_1^{n/2} \oplus \ell_\infty^{n/2}$ shows that the position of the unit ball is crucial for this statement as well.

We close this section with a variation of the minimal mean width position: Consider a symmetric convex body K in $\mathbb{R}^n$, and the problem of minimizing $M(TK)M^*(TK)$ over all $T \in \mathrm{SL}_n$. Repeating the procedure of Theorems 3.1 and 3.3 we obtain the following condition for the minimum position:

THEOREM 3.9: *Let K be a symmetric convex body in $\mathbb{R}^n$, and assume that $M(K)M^*(K) \leq M(TK)M^*(TK)$ for every $T \in \mathrm{SL}_n$. Then,*

$$\frac{1}{M}\int_{S^{n-1}} \|u\|_K\langle u,\theta\rangle^2\sigma(du) = \frac{1}{M^*}\int_{S^{n-1}} \|u\|_{K^\circ}\langle u,\theta\rangle^2\sigma(du), \tag{38}$$

for every $\theta \in S^{n-1}$.

Proof: Without loss of generality we may assume that K is smooth enough. Let $R \in L(\mathbb{R}^n)$ and $\varepsilon > 0$ be small enough, and write $T^{-1} = I + \varepsilon R$. Then, $T^* = (I+\varepsilon R^*)^{-1} = I + \sum_{k=1}^{\infty}(-1)^k\varepsilon^k(R^k)^*$, and our assumption about K takes the form

$$M(K)M^*(K) \leq \int_{S^{n-1}} \|u+\varepsilon Ru\|_K\sigma(du)\int_{S^{n-1}} \|u-\varepsilon R^*u\|_{K^\circ}\sigma(du)+O(\varepsilon^2), \tag{39}$$

which implies

$$MM^* \leq \left(M + \varepsilon\int_{S^{n-1}} \langle\nabla h_{K^\circ}(u), Ru\rangle\right)\left(M^* - \varepsilon\int_{S^{n-1}} \langle\nabla h_K(u), R^*u\rangle\right)+O(\varepsilon^2). \tag{40}$$

Letting $\varepsilon \to 0^+$ and replacing R by $-R$, we have

$$\frac{1}{M}\int_{S^{n-1}} \langle \nabla h_{K^\circ}(u), Ru\rangle \sigma(du) = \frac{1}{M^*}\int_{S^{n-1}} \langle \nabla h_K(u), R^* u\rangle \sigma(du) \tag{41}$$

for every $R \in L(\mathbb{R}^n)$. Using (40) with $R_\theta(x) = \langle x, \theta\rangle\theta$, $\theta \in S^{n-1}$, we get

$$\frac{1}{M}\int_{S^{n-1}} \langle \nabla h_{K^\circ}(u), \theta\rangle\langle u, \theta\rangle \sigma(du) = \frac{1}{M^*}\int_{S^{n-1}} \langle \nabla h_K(u), \theta\rangle\langle u, \theta\rangle \sigma(du) \tag{42}$$

for every $\theta \in S^{n-1}$. Taking into account Lemma 3.2, we conclude the proof. ∎

We do not know if (38) implies the minimality condition of Theorem 3.9 (nevertheless, we find (38) quite appealing, since it demonstrates once again the deep relation between a body and its polar).

4. Quermassintegrals and volume preserving transformations

We say that a convex body K **minimizes** W_i if $W_i(K) \leq W_i(TK)$ for every volume preserving linear transformation T. Since $nW_1(K) = \partial(K)$, a body K minimizes W_1 if and only if it has minimal surface area. Also, since $2W_{n-1}(K) = \omega_n w(K)$, a body K minimizes W_{n-1} if and only if it has minimal mean width.

Our purpose is to find necessary and sufficient conditions for a convex body K to minimize W_i, $i = 1, \dots, n-1$. We first show that such a body is a solution of a much more general problem:

PROPOSITION 4.1: *Let $i = 1, \dots, n-1$, and assume that the convex body K minimizes W_i. Then,*

$$V(T_1K, \dots, T_{n-i}K, D_n; i) \geq W_i(K) \tag{1}$$

for any $T_1, \dots, T_{n-i} \in \mathrm{SL}_n$.

Proof: We have $W_i(T_jK) \geq W_i(K)$, $j = 1, \dots, n-i$. As a consequence of the Alexandrov–Fenchel inequality we see that

$$V(T_1K, \dots, T_{n-i}K, D_n; i) \geq W_i(T_1K)^{1/n-i} \cdots W_i(T_{n-i}K)^{1/n-i}, \tag{2}$$

and this proves our claim. ∎

The arguments we used for the surface area and the mean width apply to every quermassintegral and provide necessary conditions for the minimal position:

PROPOSITION 4.2: *Assume that K is smooth enough and minimizes W_i. Then,*

$$\int_{S^{n-1}} \langle \nabla h_K(u), Ru\rangle dS_{n-i-1}(K,u) = [\mathrm{tr} R] W_i(K) \tag{3}$$

for any $R \in L(\mathbb{R}^n)$.

Proof: Let $T \in L(\mathbb{R}^n)$ and $\varepsilon > 0$ be small enough. Then, $(I+\varepsilon T)/[\det(I+\varepsilon T)]^{1/n}$ is volume preserving. Therefore,

$$[\det(I+\varepsilon T)]^{(n-i)/n} W_i(K) \leq V((I+\varepsilon T)K; n-i, D_n; i). \tag{4}$$

Since $(I+\varepsilon T)K \subseteq K + \varepsilon TK$, using the monotonicity of the mixed volumes we get

$$[\det(I+\varepsilon T)]^{(n-i)/n} W_i(K) \leq V(K+\varepsilon TK; n-i, D_n; i). \tag{5}$$

We have

$$[\det(I+\varepsilon T)]^{(n-i)/n} = 1 + \varepsilon \frac{n-i}{n} \mathrm{tr} T + O(\varepsilon^2),$$

and linearity of the mixed volumes with respect to its arguments shows that $V(K+\varepsilon TK; n-i, D_n; i) = W_i(K) + (n-i)\varepsilon V(TK, K; n-i-1, D_n; i) + O(\varepsilon^2)$. Letting $\varepsilon \to 0^+$ we see that

$$\frac{\mathrm{tr} T}{n} W_i(K) \leq V(TK, K; n-i-1, D_n; i) = \frac{1}{n} \int_{S^{n-1}} h_{TK}(u) dS_{n-i-1}(K,u). \tag{6}$$

Now, let $R \in L(\mathbb{R}^n)$ and set $T^* = I + \varepsilon R$ where $\varepsilon > 0$. Since $h_{TK}(u) = h_K(T^*u) = h_K(u+\varepsilon Ru)$, we get

$$W_i(K) + \varepsilon \frac{\mathrm{tr} R}{n} W_i(K) \leq \frac{1}{n} \int_{S^{n-1}} h_K(u+\varepsilon Ru) dS_{n-i-1}(K,u). \tag{7}$$

But, $h_K(u+\varepsilon Ru) = h_K(u) + \varepsilon \langle \nabla h_K(u), Ru\rangle + O(\varepsilon^2)$, so letting $\varepsilon \to 0^+$ and using (2.5), we have

$$\frac{\mathrm{tr} R}{n} W_i(K) \leq \frac{1}{n} \int_{S^{n-1}} \langle \nabla h_K(u), Ru\rangle dS_{n-i-1}(K,u). \tag{8}$$

Replacing R by $-R$ we get the reverse inequality, therefore

$$[\mathrm{tr} R] W_i(K) = \int_{S^{n-1}} \langle \nabla h_K(u), Ru\rangle dS_{n-i-1}(K,u) \tag{9}$$

for every $R \in L(\mathbb{R}^n)$. ∎

PROPOSITION 4.3: *Let $i = 1, \dots, n-1$. If a convex body K in $\mathbb{R}^n$ minimizes W_i, then $S_{n-i}(K, \cdot)$ is isotropic.*

Proof: Assume that K minimizes W_i. For every $U \in \mathrm{SL}_n$ we have

$$W_i(UK) = V(K; n-i, U^{-1}D_n; i) \geq W_i(K). \tag{10}$$

Let $T \in L(\mathbb{R}^n)$ and $\varepsilon > 0$ be small enough. Then, $U^{-1} = (I+\varepsilon T)/[\det(I+\varepsilon T)]^{1/n}$ is volume preserving, therefore

$$V(K; n-i, D_n + \varepsilon T D_n; i) \geq [\det(I+\varepsilon T)]^{i/n} W_i(K). \tag{11}$$

Observe that the right hand side is

$$W_i(K) + \frac{i\varepsilon \mathrm{tr} T}{n} W_i(K) + O(\varepsilon^2),$$

while the left hand side is $W_i(K) + i\varepsilon V(K; n-i, D_n; i-1, TD_n) + O(\varepsilon^2)$. Letting $\varepsilon \to 0^+$ and, taking into account (2.3), we get

$$\frac{1}{n}\int_{S^{n-1}} h_{TD_n}(u) dS_{n-i}(K,u) \geq \frac{\mathrm{tr} T}{n} W_i(K) \tag{12}$$

for every $T \in L(\mathbb{R}^n)$. Let $R \in L(\mathbb{R}^n)$ and set $T^* = I + \varepsilon R$. We have $h_{TD_n}(u) = |T^*u| = |u + \varepsilon R u| = 1 + \varepsilon\langle u, Ru\rangle + O(\varepsilon^2)$, so (12) becomes

$$\int_{S^{n-1}} \{1 + \varepsilon \langle u, Ru\rangle + O(\varepsilon^2)\} dS_{n-i}(K,u) \geq nW_i(K) + \varepsilon[\mathrm{tr} R] W_i(K). \tag{13}$$

Letting $\varepsilon \to 0^+$, using (2.6) and replacing R by $-R$ we conclude that

$$\int_{S^{n-1}} \langle u, Ru\rangle dS_{n-i}(K,u) = [\mathrm{tr} R] W_i(K) \tag{14}$$

for every $R \in L(\mathbb{R}^n)$. This shows that $S_{n-i}(K, \cdot)$ is isotropic. ∎

In order to proceed we need to introduce some terminology and notation. If A is a selfadjoint linear transformation of $\mathbb{R}^n$, we denote by $s_j(A)$ the j-th elementary symmetric function $s_j(\lambda_1, \dots, \lambda_n)$ of the eigenvalues $\lambda_1, \dots, \lambda_n$ of A:

$$s_j(A) = \sum_{1 \leq k_1 < \cdots < k_j \leq n} \lambda_{k_1} \cdots \lambda_{k_j}. \tag{15}$$

The j-th Newton operator of A is defined by

$$T_j(A) = s_j(A) I - s_{j-1}(A) A + \cdots + (-1)^j A^j. \tag{16}$$

We set $s_0(A) = 1$ and $T_0(A) = I$. We also agree that $T_j(A) = 0$ if $j < 0$.

Some known properties of $s_j(A)$ and $T_j(A)$ are listed in the Proposition below (see e.g. Reilly [Re]):

PROPOSITION 4.4: *Let $A \in L(\mathbb{R}^n)$ be selfadjoint, and assume that it has matrix (a_{kl}) with respect to some basis of $\mathbb{R}^n$. Then, $T_j(A)$ is selfadjoint and*

(i) $s_j(A) = \frac{1}{j!}\sum \delta^{k_1\cdots k_j}_{l_1\cdots l_j} a_{k_1 l_1}\cdots a_{k_j l_j}$.
(ii) $[T_j(A)]_{kl} = \frac{1}{j!}\sum \delta^{k_1\cdots k_j k}_{l_1\cdots l_j l} a_{k_1 l_1}\cdots a_{k_j l_j}$.
(iii) $\mathrm{tr}(T_j(A)\circ A) = (j+1)s_{j+1}(A)$.
(iv) $T_j(A) = s_j(A)I - T_{j-1}(A)\circ A$.
(v) $\mathrm{tr}(T_j(A)) = (n-j)s_j(A)$.

Here, we denote by $\delta^{k_1\cdots k_j}_{l_1\cdots l_j}, 1 \le j \le n$, the Kronecker symbol which has the value $+1$ (respectively, -1) if $k_1,\dots,k_j$ are distinct and $(l_1,\dots,l_j)$ is an even (respectively, odd) permutation of $(k_1,\dots,k_j)$. If not, then the symbol takes the value 0. ∎

We will also use the following consequence of Green's formula (see [Fi]):

PROPOSITION 4.5: *Let $f\colon \mathbb{R}^n\backslash\{o\} \to \mathbb{R}$ and $F\colon \mathbb{R}^n\backslash\{o\} \to \mathbb{R}^n$ be homogeneous functions of degree p and q respectively. Assume that ∇f and $\mathrm{div}F$ are continuous. Then,*

$$\begin{aligned}(17)\qquad \int_{S^{n-1}} f(u)\mathrm{div}F(u)\sigma(du) =& (p+q+n-1)\int_{S^{n-1}}\langle f(u)F(u),u\rangle\sigma(du)\\ &-\int_{S^{n-1}}\langle\nabla f(u),F(u)\rangle\sigma(du).\end{aligned}$$ ∎

Note that Lemma 3.2 is a special case of Proposition 4.5: choose $f(x) = h_K(x)$ and $F(x) = \langle x,\theta\rangle\theta$.

Let K be a convex body in $\mathbb{R}^n$, and assume that h_K is a C^3-function. For every $x \in \mathbb{R}^n\backslash\{o\}$ the Hessian $\mathcal{H}_x := (\partial^2_{kl}h_K)$ of h_K defines a selfadjoint linear transformation of $\mathbb{R}^n$. If $u \in S^{n-1}$, then $s_j(\mathcal{H}_u) = s_j(K,u)$ (for simplicity we will write $s_j(u)$). In this context, one has the following additional properties of the Newton operator $T_j(\mathcal{H}_u)$ (see [BH]):

PROPOSITION 4.6: *Assume that h_K has continuous partial derivatives of order three in $\mathbb{R}^n\backslash\{o\}$. Then,*

(i) $(j+1)s_{j+1}(x) = \mathrm{div}[(T_j(\mathcal{H}_x))(\nabla h_K(x))], \quad j = 0,\dots,n-2$.
(ii) $\mathcal{H}_x(x) = o, \quad (T_j(\mathcal{H}_x))(x) = s_j(x)x$. ∎

Combining the above results we obtain the following:

THEOREM 4.7: *Let K be a convex body in $\mathbb{R}^n$, whose support function h_K is C^3. Then, for every $j = 0,1,\dots,n-2$ and any $\theta \in S^{n-1}$, we have*

$$(18)\qquad \int_{S^{n-1}}[(n+1-j)h_K(u)s_j(u) - (j+1)s_{j+1}(u)]\langle u,\theta\rangle^2\sigma(du)$$

$$= 2\int_{S^{n-1}} \langle (T_j(\mathcal{H}_u))(\nabla h_K(u)), \theta\rangle \langle u,\theta\rangle \sigma(du).$$

Proof: Let $f(x) = \langle x,\theta\rangle^2$. By Proposition 4.6(i),

$$\int_{S^{n-1}} (j+1)s_{j+1}(u)\langle u,\theta\rangle^2\sigma(du) = \int_{S^{n-1}} f(u)\mathrm{div}[(T_j(\mathcal{H}_u)(\nabla h_K(u)))]\sigma(du). \tag{19}$$

Since f and T_j are homogeneous of degree 2 and $-j$ respectively, Proposition 4.5 shows that this last integral is equal to

$$(n+1-j)\int_{S^{n-1}} \langle T_j(\nabla h_K(u)), u\rangle\langle u,\theta\rangle^2\sigma(du) - 2\int_{S^{n-1}} \langle T_j(\nabla h_K(u)),\theta\rangle\langle u,\theta\rangle\sigma(du). \tag{20}$$

To complete the proof, observe that since T_j is selfadjoint by Proposition 4.6(ii) we have

$$\langle T_j(\nabla h_K(u)), u\rangle = \langle \nabla h_K(u), T_j(u)\rangle = s_j(u)\langle\nabla h_K(u), u\rangle = s_j(u)h_K(u). \qquad \blacksquare$$

Note first that Theorem 3.3 is a consequence of Theorem 4.7: When $j=0$, (18) takes the form

$$(n+1)\int_{S^{n-1}} h_K(u)\langle u,\theta\rangle^2\sigma(du) \tag{21}$$

$$= \int_{S^{n-1}} \langle u,\theta\rangle^2 dS_1(K,u) + 2\int_{S^{n-1}} \langle\nabla h_K(u),\theta\rangle\langle u,\theta\rangle\sigma(du).$$

By Theorem 3.1 and Proposition 4.3, the last two integrals are independent of $\theta\in S^{n-1}$, hence $\nu_K = h_K d\sigma$ is isotropic.

We now consider the case $j=1$, which corresponds to the quermassintegral W_{n-2}:

THEOREM 4.8: *Let K be a convex body in $\mathbb{R}^n$, whose support function h_K is C^3. If K minimizes W_{n-2}, then the measures $s_2(u)\sigma(du)$ and $[h_K(u)s_1(u) + |\nabla h_K(u)|^2]\sigma(du)$ are isotropic.*

Proof: We have $T_1(\mathcal{H}_u)(\nabla h_K(u)) = s_1(u)\nabla h_K(u) - \mathcal{H}_u(\nabla h_K(u))$. Then, Theorem 4.7 implies that for every $\theta\in S^{n-1}$,

$$n\int_{S^{n-1}} h_K(u)s_1(u)\langle u,\theta\rangle^2\sigma(du) + 2\int_{S^{n-1}} \langle\mathcal{H}_u(\nabla h_K(u)),\theta\rangle\langle u,\theta\rangle\sigma(du) \tag{22}$$

$$= 2\int_{S^{n-1}} \langle u,\theta\rangle^2 dS_2(K,u) + 2\int_{S^{n-1}} \langle\nabla h_K(u),\theta\rangle\langle u,\theta\rangle dS_1(K,u).$$

Assume that K minimizes W_{n-2}. By Propositions 4.2 and 4.3, the expression on the right hand side of (22) does not depend on θ. On the other hand, it is easy to check that

$$2\mathcal{H}_u(\nabla h_K(u)) = \nabla\left(|\nabla h_K(u)|^2\right). \tag{23}$$

Applying Proposition 4.5 with $F(x) = \langle x, \theta\rangle\theta$ and $f(x) = |\nabla h_K(x)|^2$, we get

$$\begin{aligned}\int_{S^{n-1}} |\nabla h_K(u)|^2\sigma(du) =& n\int_{S^{n-1}} |\nabla h_K(u)|^2\langle u,\theta\rangle^2\sigma(du)\\ &-2\int_{S^{n-1}}\langle\mathcal{H}_u(\nabla h_K(u)),\theta\rangle\langle u,\theta\rangle\sigma(du).\end{aligned} \tag{24}$$

Inserting this into (22) we see that

$$\int_{S^{n-1}} [s_1(u)h_K(u) + |\nabla h_K(u)|^2]\langle u,\theta\rangle^2\sigma(du)$$

does not depend on θ. This completes the proof. ∎

Using the same tools one can obtain analogous necessary isotropic conditions for the position which minimizes each quermassintegral. It is an interesting question to determine a set of necessary and sufficient isotropic conditions for the position minimizing W_i, $i = 2, \dots, n-2$.

5. John's Theorem

A classical result of F. John [Jo] states that $d(X, \ell_2^n) \leq \sqrt{n}$ for every n-dimensional normed space X, where ℓ_2^n is Euclidean space, and d stands for the Banach–Mazur distance. One comes up with this estimate while studying the following extremal problem:

Let K be a symmetric convex body in $\mathbb{R}^n$. Maximize $|\det T|$ over all $T : \ell_2^n \to X = X_K$ with $\|T\| = 1$.

If T_0 is a solution of this problem, then $T_0 D_n$ is the ellipsoid of maximal volume which is inscribed in K. One can easily establish existence and uniqueness of such an ellipsoid. In the spirit of our discussion, we may equivalently formulate the problem as follows:

Let K be a symmetric convex body in $\mathbb{R}^n$. Minimize $\|T : \ell_2^n \to X_K\|$ over all volume preserving transformations T.

We shall see that our standard variational argument provides all the available information about this "maximal volume ellipsoid position". In particular, one may naturally interpret the well-known John's representation of the identity as an isotropic condition.

To this end, assume that the identity map I is a solution of the problem, and normalize so that

$$(1) \qquad \|I\colon \ell_2^n \to X_K\| = 1 = \min\{\|T\colon \ell_2^n \to X_K\| : |\mathrm{det}T| = 1\}.$$

This means that the Euclidean unit ball D_n is the maximal volume ellipsoid of K. Our first result provides a necessary "trace condition" on K:

THEOREM 5.1: *Let K be a smooth enough symmetric convex body in $\mathbb{R}^n$ and assume that D_n is the maximal volume ellipsoid of K. Then, for every $T \in L(\mathbb{R}^n)$ we can find a contact point x of K and D_n such that*

$$(2) \qquad \langle x, Tx\rangle \geq \frac{\mathrm{tr}T}{n}.$$

Proof: Let $S \in L(\mathbb{R}^n)$. We shall first show that there exists a contact point x of K and D_n such that

$$(3) \qquad \|Sx\|_K \geq \frac{\mathrm{tr}S}{n}.$$

Let $\varepsilon > 0$ be small enough. From (1) we have

$$(4) \qquad \|I + \varepsilon S : \ell_2^n \to X_K\| \geq [\det(I + \varepsilon S)]^{1/n} = 1 + \varepsilon\frac{\mathrm{tr}S}{n} + O(\varepsilon^2).$$

Choose any $x_\varepsilon \in S^{n-1}$ such that $\|x_\varepsilon + \varepsilon S x_\varepsilon\|_K = \|I + \varepsilon S\|$. Since $D_n \subseteq K$, we have $\|x_\varepsilon\|_K \leq 1$. Therefore, combining (4) with the triangle inequality for $\|\cdot\|_K$ we see that

$$(5) \qquad \|Sx_\varepsilon\|_K \geq \frac{\mathrm{tr}S}{n} + O(\varepsilon).$$

By compactness, we may find $x \in S^{n-1}$ and a sequence $\varepsilon_m \to 0$ such that $x_{\varepsilon_m} \to x$. By (5) we obviously have

$$\|Sx\|_K \geq \frac{\mathrm{tr}S}{n}.$$

On the other hand,

$$(6) \qquad \|x\|_K = \lim_{m\to\infty} \|x_{\varepsilon_m} + \varepsilon_m S x_{\varepsilon_m}\|_K = \lim_{m\to\infty} \|I + \varepsilon_m S\| = \|I\| = 1.$$

This shows that x is a contact point of K and D_n, which proves (3).

Now, let $T \in L(\mathbb{R}^n)$ and write $S = I + \varepsilon T$, $\varepsilon > 0$. We can find x_ε such that $\|x_\varepsilon\|_K = |x_\varepsilon| = 1$ and

$$\|x_\varepsilon + \varepsilon T x_\varepsilon\|_K \geq \frac{\mathrm{tr}(I + \varepsilon T)}{n} = 1 + \varepsilon \frac{\mathrm{tr} T}{n}. \tag{7}$$

We write $\|x_\varepsilon + \varepsilon T x_\varepsilon\|_K = 1 + \varepsilon \langle \nabla \|x_\varepsilon\|_K, T x_\varepsilon \rangle + O(\varepsilon^2)$, and from (7) we get

$$\langle \nabla \|x_\varepsilon\|_K, T x_\varepsilon \rangle \geq \frac{\mathrm{tr} T}{n} + O(\varepsilon).$$

Choosing again $\varepsilon_m \to 0^+$ such that $x_{\varepsilon_m} \to x \in S^{n-1}$, we see that x is a contact point of K and D_n which satisfies

$$\langle \nabla \|x\|_K, T x \rangle \geq \frac{\mathrm{tr} T}{n}. \tag{8}$$

Moreover, since $\nabla \|x\|_K$ is the point on the boundary of K° at which the outer unit normal is parallel to x and x is a contact point of K and D_n, we must have $\nabla \|x\|_K = x$. This proves the theorem. ∎

From Theorem 5.1 we can easily recover all the well-known properties of the maximal volume ellipsoid:

THEOREM 5.2: *Let D_n be the maximal volume ellipsoid of K. Then, $K \subset \sqrt{n} D_n$.*

Proof: Let $x \in \mathbb{R}^n$ and consider the map $Ty = \langle y, x \rangle x$. By Theorem 5.1, we can find a contact point z of K and D_n such that

$$\langle z, Tz \rangle \geq \frac{\mathrm{tr} T}{n} = \frac{|x|^2}{n}. \tag{9}$$

But,

$$\langle z, Tz \rangle = \langle z, x \rangle^2 \leq \|z\|_{K^\circ}^2 \|x\|_K^2 = \|x\|_K^2. \tag{10}$$

Therefore, $|x| \leq \sqrt{n} \|x\|_K$. This is equivalent to the assertion of the theorem. ∎

Theorem 5.2 provides the estimate $d(X, \ell_2^n) \leq \sqrt{n}$ for the Banach–Mazur distance from an arbitrary n-dimensional normed space to ℓ_2^n. From Theorem 5.1 we can also deduce the Dvoretzky–Rogers lemma:

THEOREM 5.3: *Let D_n be the maximal volume ellipsoid of K. There exist pairwise orthogonal vectors $y_1, \dots, y_n$ in $\mathbb{R}^n$ such that*

$$\left(\frac{n-i+1}{n}\right)^{1/2} \leq \|y_i\|_K \leq |y_i| = 1, \quad i = 1, \dots, n.$$

Proof: We define the y_i's inductively. The first vector y_1 can be any of the contact points of K and D_n. Assume that $y_1, \dots, y_{i-1}$ have been defined. Let $F_i = \text{span}\{y_1, \dots, y_{i-1}\}$. Then, $\text{tr}(P_{F_i^\perp}) = n - i + 1$, and by Theorem 5.1 there exists a contact point x_i such that

$$|P_{F_i^\perp} x_i|^2 = \langle x_i, P_{F_i^\perp} x_i \rangle \geq \frac{n-i+1}{n}. \tag{11}$$

It follows that $\|P_{F_i} x_i\| \leq |P_{F_i} x_i| \leq \sqrt{(i-1)/n}$. We set $y_i = P_{F_i^\perp} x_i / |P_{F_i^\perp} x_i|$. Then,

$$1 = |y_i| \geq \|y_i\|_K \geq \langle x_i, y_i \rangle = |P_{F_i^\perp} x_i| \geq \left(\frac{n-i+1}{n}\right)^{1/2}. \tag{12}$$ ∎

Note that the argument shows that for every k-dimensional subspace F there exists a contact point x of K and D_n such that $|P_F x|^2 = \langle x, P_F x \rangle \geq k/n$.

Finally, a separation argument and Theorem 5.1 give us *John's representation of the identity*:

THEOREM 5.4: *Let D_n be the maximal volume ellipsoid of K. There exist contact points $x_1, \dots, x_m$ of K and D_n and positive real numbers $\lambda_1, \dots, \lambda_m$ such that*

$$I = \sum_{i=1}^{m} \lambda_i x_i \otimes x_i.$$

Proof: Consider the convex hull C of all operators $x \otimes x$, where x is a contact point of K and D_n. One can easily see that the assertion of the theorem is equivalent to $I/n \in C$. If this is not true, there exists $T \in L(\mathbb{R}^n)$ such that

$$\langle T, I/n \rangle > \langle x \otimes x, T \rangle \tag{13}$$

for every contact point x. But, $\langle T, I/n \rangle = \text{tr}T/n$ and $\langle x \otimes x, T \rangle = \langle x, Tx \rangle$. Therefore, (13) would contradict Theorem 5.1. ∎

Theorem 5.4 implies that

$$\sum_{i=1}^{m} \lambda_i \langle x_i, \theta \rangle^2 = 1 \tag{14}$$

for every $\theta \in S^{n-1}$. In our terminology, the measure μ on S^{n-1} that gives mass λ_i to the point x_i, $i = 1, \dots, m$, is isotropic. In this sense, John's position is an isotropic position. Conversely, following [Ba4] we have:

PROPOSITION 5.5: *Let K be a symmetric convex body in $\mathbb{R}^n$ which contains the Euclidean unit ball D_n. Assume that there exists an isotropic Borel measure μ on S^{n-1} which is supported by the contact points of K and D_n. Then, D_n is the maximal volume ellipsoid of K.*

Proof: Let $\|\mu\| = \mu(S^{n-1})$ and $A \subset S^{n-1}$ be the support of μ. Define

$$L = \{y \in \mathbb{R}^n : |\langle x, y\rangle| \leq 1, x \in A\}. \tag{15}$$

Since $K \subseteq L$, it clearly suffices to prove that D_n is the maximal volume ellipsoid of L. Let

$$E = \{y \in \mathbb{R}^n : \sum_{j=1}^{n} \alpha_j^{-2} \langle y, v_j\rangle^2 \leq 1\}, \tag{16}$$

where $\{v_j\}$ is an orthonormal basis of $\mathbb{R}^n$ and $\alpha_j > 0$. Assume that $E \subseteq L$. For every $x \in A$ we have

$$y(x) = \left(\sum_{j=1}^{n} \alpha_j^2 \langle x, v_j\rangle^2\right)^{-1/2} \sum_{j=1}^{n} \alpha_j^2 \langle x, v_j\rangle v_j \in E \subseteq L, \tag{17}$$

hence, $|\langle x, y(x)\rangle| \leq 1$ gives

$$\sum_{j=1}^{n} \alpha_j^2 \langle x, v_j\rangle^2 \leq 1, \quad x \in A. \tag{18}$$

Our hypotheses imply that

$$\begin{aligned} \sum_{j=1}^{n} \alpha_j &= \sum_{j=1}^{n} \alpha_j \frac{n}{\|\mu\|} \int_{S^{n-1}} \langle x, v_j\rangle^2 \mu(dx) \\ &= \frac{n}{\|\mu\|} \int_{S^{n-1}} \sum_{j=1}^{n} \alpha_j \langle x, v_j\rangle^2 \mu(dx). \end{aligned} \tag{19}$$

Using (18) and the Cauchy-Schwarz inequality we see that

$$\sum_{j=1}^{n} \alpha_j \langle x, v_j\rangle^2 \leq \left(\sum_{j=1}^{n} \alpha_j^2 \langle x, v_j\rangle^2\right)^{1/2} \left(\sum_{j=1}^{n} \langle x, v_j\rangle^2\right)^{1/2} \leq 1 \tag{20}$$

for every $x \in A$. Then, (19) becomes

$$\sum_{j=1}^{n} \alpha_j \leq n. \tag{21}$$

By the arithmetic–geometric means inequality we get $\prod \alpha_j \leq 1$. That is $|E| \leq |D_n|$. Moreover, we can have equality only if all α_j's are equal to 1, which shows that D_n is the unique maximal volume ellipsoid of L. ∎

Theorem 5.4 and Proposition 5.5 provide the following characterization of John's position:

"Let K be a symmetric convex body in $\mathbb{R}^n$ which contains the Euclidean unit ball D_n. Then, D_n is the maximal volume ellipsoid of K if and only if there exists an isotropic measure μ supported by the contact points of K and D_n."

Let us discuss one more problem of the same nature: Let K be a symmetric convex body in $\mathbb{R}^n$ and $\|\cdot\|$ be the corresponding norm. Assume that $(1/a)|x| \leq \|x\| \leq b|x|$ for every $x \in \mathbb{R}^n$. It is clear that $M(K)a(K) \geq 1$, and we are interested in

$$\min\{M(TK) |\, T \in \mathrm{GL}_n, a(TK) = 1\}. \tag{22}$$

The condition $a(TK) = 1$ means that $TK \subseteq D_n$ but there exist contact points of TK and D_n. We then have the following condition for the minimum position:

THEOREM 5.6: *Let K be a symmetric convex body in $\mathbb{R}^n$ satisfying $a(K) = 1$ and $M(K) \leq M(TK)$ for every $T \in \mathrm{GL}_n$ with $a(TK) = 1$. Then, for every $\theta \in S^{n-1}$ we can find contact points x_1, x_2 of K and D_n such that*

$$1 + \langle x_1, \theta\rangle^2 \leq \frac{n+1}{M} \int_{S^{n-1}} \|u\|_K \langle u, \theta\rangle^2 \sigma(du) \leq 1 + \langle x_2, \theta\rangle^2. \tag{23}$$

Proof: Let $T \in L(\mathbb{R}^n)$ and $\varepsilon > 0$ be small enough. Then

$$T_1 := (\min_{S^{n-1}} \|x + \varepsilon Tx\|)(I + \varepsilon T)^{-1}$$

satisfies $a(T_1K) = 1$. Therefore,

$$\int_{S^{n-1}} \|u + \varepsilon Tu\| \sigma(du) \geq M(K) \min_{x \in S^{n-1}} \|x + \varepsilon Tx\|. \tag{24}$$

If we write $\|u + \varepsilon Tu\| = \|u\| + \varepsilon \langle \nabla h_{K^\circ}(u), Tu\rangle + O(\varepsilon^2)$, we see that

$$\int_{S^{n-1}} \langle \nabla h_{K^\circ}(u), Tu\rangle \sigma(du) + O(\varepsilon) \geq M(K) \frac{\min_{S^{n-1}} \|x + \varepsilon Tx\| - 1}{\varepsilon}. \tag{25}$$

Let x_ε be a point on S^{n-1} at which the minimum is attained. If x is a contact point of K and D_n, we must have $1+\varepsilon\|T\| \geq \|x+\varepsilon Tx\| \geq \|x_\varepsilon+\varepsilon Tx_\varepsilon\| \geq \|x_\varepsilon\| - \varepsilon\|T\|$, where $\|T\| := \|T : \ell_2^n \to X_K\|$. It follows that

$$1 \leq \|x_\varepsilon\| \leq 1+2\varepsilon\|T\|. \tag{26}$$

Since $x_\varepsilon \in S^{n-1}$ and $\|\cdot\| \geq |\cdot|$, (25) takes the form

$$\begin{aligned}\int_{S^{n-1}} \langle \nabla h_{K^\circ}(u), Tu\rangle \sigma(du) + O(\varepsilon) &\geq M(K)\frac{|x_\varepsilon+\varepsilon Tx_\varepsilon|-1}{\varepsilon}\\ &= M(K)[\langle x_\varepsilon, Tx_\varepsilon\rangle + O(\varepsilon)].\end{aligned} \tag{27}$$

Now, we can find a sequence $\varepsilon_m \to 0$ and a point $x \in S^{n-1}$ such that $x_{\varepsilon_m} \to x$. Letting $m \to \infty$ in (27), we obtain

$$\int_{S^{n-1}} \langle \nabla h_{K^\circ}(u), Tu\rangle \sigma(du) \geq M(K)\langle x, Tx\rangle. \tag{28}$$

Also, $x \in S^{n-1}$ and using (26) we see that $\|x\| = \lim_m \|x_{\varepsilon_m}\| = 1$. That is, x is a contact point of K and D_n. Replacing T by $-T$ we find another contact point x' of K and D_n such that

$$\int_{S^{n-1}} \langle \nabla h_{K^\circ}(u), Tu\rangle \sigma(du) \leq M(K)\langle x', Tx'\rangle. \tag{29}$$

Choosing $T_\theta(x) = \langle x, \theta\rangle\theta$, $\theta \in S^{n-1}$, and applying Lemma 3.2, we obtain (23). ∎

The condition of the Theorem shows in a sense that the minimum position of the problem is rich in contact points with the circumscribed ball. The dual problem of maximizing M under the condition $b=1$ has exactly the same answer.

6. Minimal surface area and M-position

If K and L are convex bodies in $\mathbb{R}^n$, we write $N(K,L)$ for the *covering number* of K by L (that is, the minimum number of translates of L whose union covers K). If $|K| = |D_n|$, we say that K is in M-position (with parameter $\delta > 0$) if

$$N(K, D_n) \leq \exp(\delta n). \tag{1}$$

One can then prove (see [MP2] for the non-symmetric case) that

$$N(K,D_n)\cdot N(D_n,K)\cdot N(K^\circ,D_n)\cdot N(D_n,K^\circ) \leq \exp(\delta_1 n), \tag{2}$$

where $\delta_1 = c\delta$, and $c > 0$ is an absolute constant. Moreover, condition (1) is equivalent to

$$|K + D_n|^{1/n} \leq c|D_n|^{1/n}. \tag{3}$$

This isomorphically defined position is the best representative of the affine class of a body in volume computations: this is mainly due to the fact that reverse Brunn–Minkowski inequalities hold for bodies in M-position [M2].

We define a function $f\colon [0, +\infty) \to \mathbb{R}$ by

$$f(t) = \min\{|TK + tD_n| \mid T \in \mathrm{SL}_n\}. \tag{4}$$

For every $t > 0$ there exists a volume preserving T_t such that $|T_tK + tD_n| = f(t)$. It is clear that UT_t has the same property for every $U \in O(n)$. By (3) we see that T_1K is in M-position. This suggests that M-position can be described as the solution of a minimum problem similar to the ones we discussed in the previous sections.

We start with the following observation:

Lemma 6.1: *Let K be a convex body in $\mathbb{R}^n$. Then,*

$$|K + tA_1D_n + sA_2D_n| \geq \min\{|K + (t+s)A_1D_n|, |K + (t+s)A_2D_n|\} \tag{5}$$

for every $A_1, A_2 \in \mathrm{GL}_n$ and $t, s > 0$.

Proof: It is an immediate consequence of the Brunn–Minkowski inequality, since

$$K+tA_1D_n+sA_2D_n \supseteq \frac{t}{t+s}(K + (t+s)A_1D_n)+\frac{s}{t+s}(K + (t+s)A_2D_n). \tag{6}$$ ∎

Theorem 6.2: *Let K be a convex body in $\mathbb{R}^n$. Assume that*

$$|K + tD_n| = f(t) \tag{7}$$

for some $t > 0$. Then, $K + tD_n$ has minimal surface area.

Proof: Let $T \in \mathrm{SL}_n$. From Steiner's formula we see that

$$\begin{aligned} &|T(K + (t-\varepsilon)D_n) + \varepsilon D_n| - |T(K + (t-\varepsilon)D_n)| \\ &= n\varepsilon W_1(T(K + (t-\varepsilon)D_n)) + O(\varepsilon^2). \end{aligned} \tag{8}$$

By the continuity of W_1 with respect to the Hausdorff metric,

$$(9)\ \partial(T(K+tD_n)) = nW_1(T(K+tD_n)) = n \lim_{\varepsilon\to 0^+} W_1(T(K+(t-\varepsilon)D_n))$$
$$= \lim_{\varepsilon\to 0^+} \frac{|T(K+(t-\varepsilon)D_n)+\varepsilon D_n| - |T(K+(t-\varepsilon)D_n)|}{\varepsilon}$$
$$= \lim_{\varepsilon\to 0^+} \frac{|K+(t-\varepsilon)D_n+\varepsilon T^{-1}D_n| - |K+(t-\varepsilon)D_n|}{\varepsilon}.$$

Since $|K+tD_n| = f(t)$, Lemma 6.1 implies that $|K+(t-\varepsilon)D_n+\varepsilon T^{-1}D_n| \geq |K+tD_n|$. Hence,

$$(10)\qquad \partial(T(K+tD_n)) \geq \lim_{\varepsilon\to 0^+} \frac{|K+tD_n| - |K+(t-\varepsilon)D_n|}{\varepsilon} = \partial(K+tD_n).$$

This shows that $K+tD_n$ has minimal surface area. ∎

Remark: It is not hard to show that

$$(11)\qquad f'(t) = \partial(T_tK+tD_n)$$

for every $t>0$. It follows that for every $t>s>0$ we have

$$(12)\qquad \int_s^t \partial(T_xK+xD_n)dx \geq \int_s^t \partial(T_tK+xD_n)dx,$$

with equality if $s=0$.

In the planar case, a convex body K has minimal perimeter (surface area) if and only if it has minimal mean width. Since $|T_tK+tD_n| = f(t)$, Theorem 6.2 shows that T_tK+tD_n has minimal mean width and, using Corollary 3.4(ii), we see that T_tK has minimal mean width. Moreover, T_t is constant up to an orthogonal transformation. That is, the solution of Problem (4) is the minimal mean width position, independently of $t>0$:

COROLLARY 6.3: *A convex body K in $\mathbb{R}^2$ satisfies $|K+tD_n| \leq |TK+tD_n|$ for every $T \in \mathrm{SL}_n$ and every $t>0$ if and only if it has minimal mean width.* ∎

It would be interesting to see if the minimal surface area position is an M-position in higher dimensions. This would provide an isometric description of the M-position. Observe that, by Theorem 6.2, the limit of T_tK as $t \to 0^+$ is the minimal surface position and, by Steiner's formula, the limit of T_tK as $t \to +\infty$ is the minimal mean width position.

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